AP Calculus AB — Quiz 0920 SOLUTIONS

Dr. Paul L. Bailey — Friday, September 20, 2024

**Problem 1.** Consider the function
$$f(x) = \frac{x^2 - 9}{x^2 - 8x + 15}.$$
Find the following limits.

**(a)** $\lim_{x\to 0} f(x)$

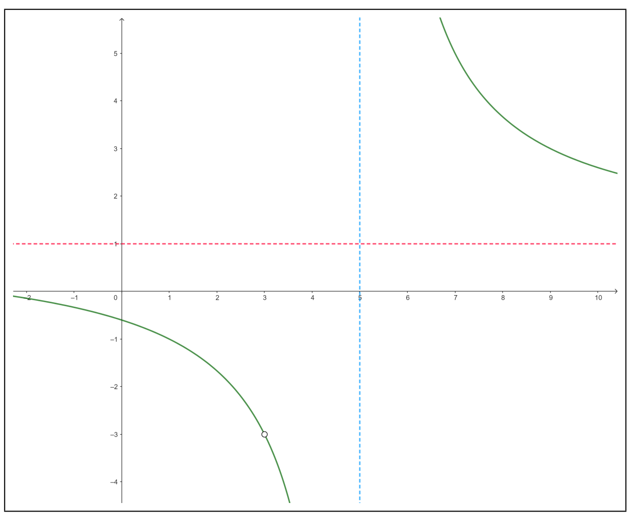
**(b)** $\lim_{x\to 3} f(x)$

**(c)** $\lim_{x\to -3} f(x)$

**(d)** $\lim_{x\to 5^+} f(x)$

**(e)** $\lim_{x\to \infty} f(x)$

*Solution.* The first step is to factor the numerator and the denominator:
$$f(x) = \frac{x^2 - 9}{x^2 - 8x + 15} = \frac{(x-3)(x+3)}{(x-3)(x-5)} = \frac{x+3}{x-5}.$$
The domain of this function is $\mathbb{R} \smallsetminus \{3, 5\}$, and the last equal sign is true on this domain. Thus $f$ has a zero at $-3$, a pole at 5, and a hole at 3. The $y$-intercept is $(0, -\frac{3}{5})$, the $x$-intercept is $(-3, 0)$, the vertical asymptote is $x = 5$, the horizontal asymptote is $y = 1$. To find the hole, plug in 3 to the last expression to get
$$\left.\frac{x+3}{x-5}\right|_{x=3} = \frac{3+3}{3-5} = -3.$$
So the hole is at $(3, -3)$. The graph is



**(a)** $\lim_{x\to 0} f(x) = \frac{3}{5}$ (plug in 0)

**(b)** $\lim_{x\to 3} f(x) = -3$ (plug 3 into the simplified form)

**(c)** $\lim_{x\to -3} f(x) = 0$ (plug in $-3$)

**(d)** $\lim_{x\to 5^+} f(x) = +\infty$ (approach the pole from the right)

**(e)** $\lim_{x\to \infty} f(x) = 1$ (the ratio of the leading coefficients)

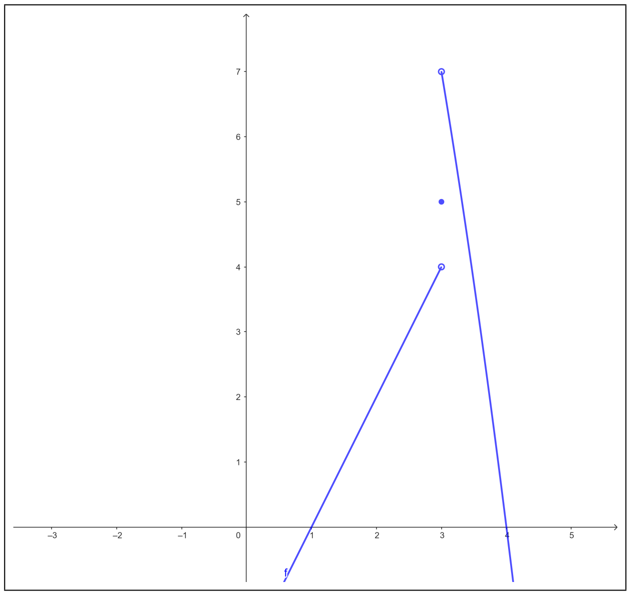
□

**Problem 2.** Let

$$f(x) = \begin{cases} 2x - 2 & \text{for } x < 3 \\ 5 & \text{for } x = 3 \\ 16 - x^2 & \text{for } x > 3 \end{cases}$$

Find $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.

*Solution.* The graph is




The left limit is gotten by plugging $x = 3$ into $2x - 2$ to get

$$\lim_{x \to 3^-} f(x) = 2x - 2|_{x=3} = 6 - 2 = 4.$$

The right limit is gotten by plugging $x = 3$ into $16 - x^2$ to get

$$\lim_{x \to 3^+} f(x) = 16 - x^2|_{x=3} = 16 - 3^2 = 7.$$

□

**Problem 3.** Compute $\lim_{x \to 0} \frac{2\tan(3x)}{5x}$.

*Solution.* We have seen that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1,$$

Thus

$$\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x \to 0} \left( \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} \right) = \left( \lim_{x \to 0} \frac{\sin(x)}{x} \right) \cdot \left( \lim_{x \to 0} \frac{1}{\cos(x)} \right) = 1 \cdot 1 = 1.$$

Therefore

$$\lim_{x \to 0} \frac{2\tan(3x)}{5x} = \frac{2}{5} \lim_{x \to 0} \frac{\tan(3x)}{x} = \frac{6}{5} \lim_{x \to 0} \frac{\tan(3x)}{3x} = \frac{6}{5} \lim_{3x \to 0} \frac{\tan(3x)}{3x} = \frac{6}{5} \cdot 1 = \frac{6}{5}.$$

□